SMOOTH MANIFOLDS FALL 2022 - HOMEWORK 8

Problem 1. Find an area form ω on S^2 such that for any rotation about some axis in $\mathbb{R}^3$, $R : S^2 \to S^2$, we have that $R^*\omega = \omega$. Prove the invariance property, and show that this volume form is unique up to positive scalar multiple.

[*Remarks*: You may use the natural coordinates for the tangent spaces as subspaces of $\mathbb{R}^3$ (ie, you may write down a form on $\mathbb{R}^3$ and restrict it to S^2 to construct the form). You may use the fact that rotations around the coordinate axes generate the group of all rotations.]

Problem 2.

(1) Prove the following convenient formula for the exterior derivative of a 1-form α, where X and Y are vector fields on M. Note that if $f \in C^\infty(M)$, then $X \cdot f$ denotes the derivative of f along the vector field X.

$$d\alpha(X, Y) = X \cdot (\alpha(Y)) - Y \cdot (\alpha(X)) - \alpha([X, Y])$$

[*Hint*: Any 1-form is locally a linear combination of forms of the form $u\,dv$ for $u, v \in C^\infty$. Evaluate both side of the desired equality for such forms.]

(2) Use this to find a formula for $\mathcal{L}_X\alpha(Y)$, where X and Y are C^∞ vector fields and α is a 1-form on M. Think magically!

Problem 3. Fix a $2n$-dimensional manifold M. Recall that a 2-form ω is called *symplectic* if $d\omega = 0$ and the n-fold wedge product of ω is a volume form on M.

(1) Show that a closed 2-form ω is symplectic if and only if for every $x \in M$ and nonzero vector $X \in T_xM$, there exists $Y \in T_xM$ such that $\omega(X, Y) \neq 0$. [*Hint*: One direction is an easy calculation by contradiction using the operator ι_X and its formula for wedge products. For the other direction, assume that you have found linearly independent vectors $X_1, Y_1, \ldots, X_k, Y_k \in T_xM$ such that $\omega(X_i, Y_i) = 1$ for every i, and all other pairwise combinations evaluate to 0. Define:

$$\phi_x^{(k)} : T_xM \to T_xM \qquad \phi_x^{(k)}(v) = v - \left(\sum_{i=1}^{k} \omega(v, X_i)Y_i - \omega(v, Y_i)X_i\right)$$

Choose $\tilde{X}_{k+1}, \tilde{Y}_{k+1}$ to be a pair such that $\tilde{X}_{k+1}$ is not in the span of the $\{X_i, Y_i\}_{i=1}^k$ and $\omega(\tilde{X}_{k+1}, \tilde{Y}_{k+1}) = 1$, and let $X_{k+1} = \phi_x^{(k)}(\tilde{X}_{k+1})$, $Y_{k+1} = \phi_x^{(k)}(\tilde{Y}_{k+1})$. Show that X_{k+1} and Y_{k+1} extend the desired properties.]

(2) Show that if α is any 1-form on M, there exists a unique vector field X_α such that $\iota_{X_\alpha}\omega = \alpha$. [*Hint*: You don't have to show regularity, and you can build the vector field pointwise by showing that the map $F_x : T_xM \to T_x^*M$ defined by $F_x(v) = \iota_v\omega(x)$ is an isomorphism of vector spaces]